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部分性觀察馬可夫決策過程下檢查—修復—重置政策

Inspection-Repair-Replacement Policy under Partially Observable Markov Decision Processes

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摘　要

本研究探討部分性觀察馬可夫決策過程，於離散性無限時間幅度下之最佳檢查—修復—重置政策。首先，假設狀態空間皆為可數有限，且其特定行動空間為：「不採取行動、期初檢查、期初瞬時修復與重置」。再者，實施檢查需耗費成本，唯可確知系統實際狀態；然除重置行動外，採取修復行動，並未能確保系統回復至最佳狀態。據此，建構最大總期望折現報酬遞迴式，在隨機優勢與單調概似率等偏序，以及二階全正條件下，推證各項重要結果；顯示最佳政策結構，具有將狀態機率向量空間，分割為至多五區域之特性。其次，建構比較互異行動空間「不採取行動、期末瞬時檢查與修復、期初瞬時重置」與「不採取行動、期初檢查、期末瞬時修復與重置」等模式之重要結果。最後，彙述後續相關研究供參考。

關鍵詞：部分性觀察馬可夫決策過程、隨機優勢、單調概似率、二階全正、檢查—修復—重置政策。

ABSTRACT

This study examines optimal inspection-repair-replacement policy for the discrete-time partially observable Markov decision processes over an infinite horizon in which the state space is finite and the action space consist of "no action, inspection at beginning, instantaneous repair and replacement at beginning." Upon inspection to determine the precise state of the system, an additional cost is required. It is noted that repair cannot return the system to an as-good-as-new state. First, we construct the recursion to maximize the expected total discounted reward. Useful results are derived under the conditions of partial orders, namely stochastic dominance and monotone likelihood ratio, as well as the totally positive of order two. Consequently, we show that the optimal policies have the structure which break up the space of state probability vectors into at most five-region. Next, alternate modeling results are set forth within two different action spaces: "no action, instantaneous inspection and repair at end, instantaneous replacement at beginning"; "no action, inpection at beginning, instantaneous repair and replacement at end." Finally, several relevant studies are presented for further consideration.

Keywords: partially observable markov decision processes (POMDP's), stochastic dominance, monotone likelihood ratio, totally positive of order two, inspection-repair-replacement policy.

I. INTRODUCTION

The problems of Markov decision processes (MDP's) subject to deterioration have been addressed by considerable literatures (for surveys and its applications see Barlow et al. (1965), McCall (1965), Pierskalla and Voelker (1976), Sherif and Smith (1981), Heyman and Sobel (1984), White (1985), White and White (1989)). For an excellent overview of partially observable Markov decision processes (POMDP's), see Monahan (1982). POMDP is a generalization of a Markov decision process which permits uncertainty regarding the state of a Markov process and allows for state information acquisition. Later White and Scherer (1989), and White (1991) integrate several computational algorithms for the POMDP's problems. Note that the POMDP can be converted into an equivalent of a completely observed MDP, following Åström (1965), Sondik (1978), Sawaki (1963), Hernández-Lerma and Marcus (1987), Bertsekas (1976).

Results characterizing the optimal policies for POMDP's differ depending on the specific assumptions of the model. The ideal conclusions are for the case where it is assumed that the underlying state of the system can be completely determined only through costly inspection-repair (see Klein (1962), Ross (1971), Smallwood and Sondik (1973), Kander and Raviv (1974), Ehrenfeld (1976), Rosenfield (1976a,b), White (1977,1978), Luss (1976, 1983), Sernik and Marcus (1991a,b)). Moreover, most models assume that a repaired-replaced system often returns to state which is as-good-as-new. In actual case, many systems subject to deterioration, including military equipment, electronic products, and manufacturing systems, are impossible to resume to the as-good-as-new state despite repair.

Summarizing results from previous related studies, we see that general models with inspection, repair and replacement options have not yet been solved. Hence, we relax this assumption and presume that the action space composing "no action, inspection, repair and replacement," bring forward the construction of the recursion for the optimal value function.

This paper adopts stochastic dominance ($\geq_{st}$) and monotone likelihood ratio ($\geq_{mlr}$) partial orders, which are presented in Karlin and Rinott (1980), Lovejoy (1986,1987) and Fernández-Gaucherand et al. (1991) for POMDP's problems on the order of the state probability vectors. Totally positive of order two (TP_2) in Albright (1979) also applies to identify the preferred ranks for the system deterioration. Furthermore, under some reasonable conditions and relevant properties, such as isotone (antitone) function discussed in Topkis (1978) and Lovejoy (1985), we are able to arrive at the optimal structural results, similar to those obtained by Ohnishi et al. (1986), and Wu and Chen (1991).

This paper is organized as follows: Section 2 describes the model in detail and states several definitions; Section 3 provides some assumptions; Section 4 formulates the recursion of the optimal inspection-repair-replacement problem under partial observation over finite and infinite horizons. In Section 5, well-known results for the optimal value function and optimal policies are derived. Section 6 describes the alternate modeling results. The paper concludes with a discussion on potential issues of the model.

II. MODEL DESCRIPTION AND DEFINITIONS

We consider the discrete-time Markov decision processes $\{X_t,\ t = 0, 1, 2, \ldots\}$, having state space S where $S = \{1, 2, \ldots, n\}$. Here, 1 represents the best state and n the worst state. Given two states i and j, $i < j$ means that the perfor-mace of state j is worse than that of state i. We also let $r = (r_1, \ldots, r_i, \ldots, r_n)$ and r stands for the reward vector, where r_i denotes the immediate reward in state i which is nonincreasing in state number. It is assumed the system undergoing deterioration is monitored imperfectly at each epoch $t, t = 0, 1, 2, \ldots$.

For convenience, we first define the space of state probability vectors $\prod(S)$, where $\prod(S) = \{\pi \in R^n : 0 \leq \pi_i \leq 1, \sum_{i=1}^{n} \pi_i = 1\}$. Hence, the only available information about the state space to the decision maker is to specify a probability vector π with elements π_i, where $\pi_i = P\{X_t = i\}$

The decision maker has four alternatives at every time period from the action space $A = \{0, 1, 2, 3\}$ and $a \in A$, where 0,1,2,3 denote the decision for no action, inspection at beginning, instantaneous repair and replacement at beginning, respectively. It is assumed that an inspection cost, denoted by I, is accrued to be independent of the underlying state. After inspection we can identify the precise state of the system. We also assume that only when replacement action is taken in state i with state-dependent cost C_i, can the system be restored to an as-good-as-new state. Due to the fact that repair does not restore the system to the best state from its current state i, but probably to a certain better state $q, q \leq i$. The repair cost of returning to the state q is M_i^q. Then, let $M^q = (0, \ldots, 0, M_q^q, \ldots, M_i^q, \ldots, M_n^q)$, and M^q denotes the repair cost vector.

Furthermore, we let transition matrix under "action a" be $P^a = [p_{ij}^a]_{n \times n}$, where p_{ij}^a represents the transition probability in state i and action a is performed, then it moves to state j. Specifically, P^0 is the Markovian deterioration matrix where no action is taken. For each decision epoch, the state of the system is observed partially by some monitoring mechanism, a finite number of observations is received. Let $O = \{1, 2, \ldots, \omega\}$, and O represents the observation space. We also let the observation matrix be $Q^a = [q_{jk}^a]_{n \times \omega}$ where the q_{jk}^a represents the conditional probability of observation, $Y_{t+1} = k$, given that $X_{t+1} = j$ and action a is performed. Here $\{Y_t \in O, t = 1, 2, \ldots\}$ is the observation process. Finally, we also let $\pi P^a = \Big((\pi P^a)_1, (\pi P^a)_2, \ldots, (\pi P^a)_n\Big) \in \prod(S)$, and πP^a designates the one-step transition probability vector that state probability vector is π and action a is taken. The controllability of observation processes depends on the choice action a in each period, and is related by the following known conditional probabilities:

$$\sigma(\pi, a) = \Big(\sigma(1|\pi, a), \sigma(2|\pi, a), \ldots, \sigma(\omega|\pi, a)\Big),$$

and

$$\theta(\pi, a, k) = \Big(\theta(1|\pi, a, k), \theta(2|\pi, a, k), \ldots, \theta(n|\pi, a, k)\Big)$$

where $\sigma(k|\pi, a) = (\pi P^a Q^0)_k$ and $\theta(j|\pi, a, k) = \dfrac{(\pi P^a)_j q^0_{jk}}{(\pi P^a Q^0)_k}$ for all $\pi \in \prod(S)$, and $a \in A - \{1\}$.

To construct an optimal policy over an infinite horizon, we let $v_t^T(\pi)$ denote the optimal expected total discounted reward in periods t through T, discounted back to the beginning of period t, given the state probability vector π. The problem is to obtain an inspection-repair-replacement policy maximizing the expected total discounted reward with $\beta \in [0, 1]$ over infinite horizon.

Before considering the assumptions of the model, we need to define the following

Definition 2.1

$F^n \equiv \{f \in R^n : f_1 \geq f_2 \geq \cdots \geq f_n\}$, where $f = (f_1, f_2, \ldots, f_n)'$. That is, F^n denotes the set of all n dimensional vectors with nonincreasing components.

We then define the following two partial orders:

Definition 2.2

Given π and $\pi' \in \prod(S)$, we say that

$$\pi \geq_{st} \pi' \text{ if } \sum_{i=1}^{k} \pi_i \geq \sum_{i=1}^{k} \pi'_i \text{ for } i, k \in S;$$

and

$$\pi \geq_{mlr} \pi' \text{ if } i \leq j \text{ implies } \pi_i \pi'_j \geq \pi_j \pi'_i \text{ for } i, j \in S,$$

where $\geq_{st}$ and $\geq_{mlr}$ represent the partial orders of stochastic dominance and monotone likelihood ratio respectively.

Definition 2.3

P^a is said to be the totally positive of order two (TP_2), if

$$\begin{vmatrix} p^a_{ij} & p^a_{il} \\ p^a_{kj} & p^a_{kl} \end{vmatrix} \geq 0 \text{ for } k \geq i, l \geq j, i, j, k, l \in S, a \in A - \{2, 3\}.$$

Definition 2.4

Letting $\left(\prod(S), \geq_{mlr}\right)$ be a partially ordered space, we say that Φ : $\prod(S) \times A \times R \rightarrow R$ has isotone differences on $\left(\prod(S), \leq_{mlr}\right)$, if $\Phi(\pi, a, v) - \Phi(\pi, a', v) \geq \Phi(\pi', a, v) - \Phi(\pi', a', v')$; and $\Phi : \prod(S) \times A \times R \rightarrow R$ has antitone differences on $\left(\prod(S), \leq_{mlr}\right)$, if $\Phi(\pi, a, v) - \Phi(\pi, a', v) \leq \Phi(\pi', a, v) - \Phi(\pi', a', v)$ for $\pi \geq_{mlr} \pi', \pi, \pi' \in \left(\prod(S), \geq_{mlr}\right)$, and $a \geq a', a, a' \in A$.

III. ASSUMPTIONS

The basic assumptions are stated below:

(A1) P^a is a $n \times n$ Markovian deterioration matrix for $P^a_{ii} < 1, i \in S - \{n\}$ and $P^a_{ij} = 0,\ j < i$ where $a \in A - \{2, 3\}$. It implies that P^0 and P^1 are upper triangular. We also assume that P^0 and P^1 have TP_2 property.

(A2) Q^a is a $n \times \omega$ TP_2 matrix for $a \in A - \{1\}$, and $Q^0 = Q^2 = Q^3$, i.e., $q^0_{jk} = q^2_{jk} = q^3_{jk}$ for $j \in S, k \in O$.

(A3) $-M^q \in F^n$.

(A4) $-C \in F^n$.

(A5) $R + M^q \in F^n$.

(A6) $R + C \in F^n$.

(A7) $-M^q + C \in F^n$.

(A8) $-C + M^q \in F^n$.

(A1) claims that the system will deteriorate when no action or inspection is performed, and it has the IFR property (see Derman (1963) and Rosenfield (1976a), in which a straightforward result is that TP_2 implies IFR). The deterioration will accelerate in its worse state. We also consider the repair or replacement action at the beginning so as to improve the current state or to restore to the best state. (A2) implies that the observation matrix is independent of the actions, excluding inspection, after which the better state gives rise to better observations. (A3) and (A4) state that repair and replacement costs are nondecreasing in state. (A5) and (A6) assure that the performance of repair and replacement are nondecreasing in state. (A7) shows the gaps between the replacement and repair costs tend to get smaller at each deterioration state. In another words, the increasing rate of the repair costs becomes greater than that of the replacement costs. Lastly, the opposite condition in (A7) is illustrated by (A8).

IV. MODEL FORMULATION

Let Φ: $\prod(S) \times A \times R \rightarrow R$, we then establish the following recursion:

$$
\begin{aligned}
v_t^T(\pi) &= \max_{a \in A}\{\Phi(\pi, a, v_{t+1}^T)\} \\
&= \max \begin{cases}
\pi r + \beta \sum_{k=1}^{\omega} \sigma(k|\pi,0) v_{t+1}^T\Big(\theta(\pi,0,k)\Big) & ,\text{a=0} \\
\pi r - I + \beta \sum_{i=1}^{n} (\pi P^0)_i v_{t+1}^T(e^i) & ,\text{a=1} \\
\pi(-M^q) + r_q + \beta \sum_{k=1}^{\omega} \sigma(k|\pi,2) v_{t+1}^T\Big(\theta(\pi,2,k)\Big) & ,\text{a=2} \\
\pi(-C) + r_1 + \beta \sum_{k=1}^{\omega} \sigma(k|\pi,3) v_{t+1}^T\Big(\theta(\pi,3,k)\Big) & ,\text{a=3}
\end{cases}
\end{aligned}
\tag{4.1}
$$

for $t = 1, 2, \ldots, T-1$, and $v_T^T(\pi) = 0$ for all $\pi \in \prod(S)$.

We let $v_t^T(\pi) = v(\pi)$ as $T \to \infty$. The new recursion is given by the following.

$$v(\pi) = \max_{a \in A}\{\Phi(\pi, a, v)\}$$
$$= \max \begin{cases} \pi r + \beta \sum_{k=1}^{\omega} \sigma(k|\pi, 0) v\Big(\theta(\pi, 0, k)\Big) & ,\text{a=0} \\ \pi r - I + \sum_{i=1}^{n} (\pi P^0)_i v(e^i) & ,\text{a=1} \\ \pi(-M^q) + r_q + \beta \sum_{k=1}^{\omega} \sigma(k|\pi, 2) v\Big(\theta(\pi, 2, k)\Big) & ,\text{a=2} \\ \pi(-C) + r_1 + \beta \sum_{k=1}^{\omega} \sigma(k|\pi, 3) v\Big(\theta(\pi, 3, k)\Big) & ,\text{a=3} \end{cases} \quad (4.2)$$

Since we assume that the state levels for system deterioration can be observed through inspections, returned to a better state after repair, or reseted to an as-good-as-new state by replacement, then $v(\pi)$ becomes

$$v(\pi) = \max_{a \in A}\{\Phi(\pi, a, v)\}$$
$$= \max \begin{cases} \pi r + \beta \sum_{k=1}^{\omega} \sigma(k|\pi, 0) v\Big(\theta(\pi, 0, k)\Big) & ,\text{a=0} \\ \pi r - I + \sum_{i=1}^{n} (\pi P^0)_i v(e^i) & ,\text{a=1} \\ \pi(-M^q) + r_q + \beta \sum_{k=1}^{\omega} \sigma(k|e^q, 0) v\Big(\theta(e^q, 0, k)\Big) & ,\text{a=2} \\ \pi(-C) + r_1 + \beta \sum_{k=1}^{\omega} \sigma(k|e^1, 0) v\Big(\theta(e^1, 0, k)\Big) & ,\text{a=3} \end{cases} \quad (4.3)$$

for all $\pi \in \prod(S)$, where e^q denotes the q_{th} element of state probability vector which is 1.

V. STRUCTURAL RESULTS

To study the characteristics of $\Phi(\pi, a, v)$ and $v(\pi)$, we need the following Lemmas.

Lemma 5.1 If $\pi \geq_{mlr} \pi'$, then $\pi \geq_{st} \pi'$ for any $\pi, \pi' \in \prod(S)$.

Proof: Refer to Lemma 5.1 in Fernández-Gaucherand et al. (1991). □

Lemma 5.2 If A is a $k \times l$ TP_2 matrix and B is a $l \times n$ TP_2 matrix, then the product AB is a $k \times n$ TP_2 matrix.

Proof: It is identical to Lemma 4.3 in Ohnishi et al. (1989), and so the proof is omitted. □

Under assumptiom (1) and for any $\pi, \pi' \in \left(\prod(S), \geq_{mlr}\right)$, all of the following Lemmas hold.

Lemma 5.3 If $\pi \geq_{mlr} \pi'$, then $\pi P^a \geq_{mlr} \pi' P^a$.

Proof: Since $\pi \geq_{mlr} \pi'$ implies that $\binom{\pi}{\pi'}$ is TP_2 from the definitions (2.2) and (2.3), it follows from Lemma 5.2 that the product $\binom{\pi}{\pi'} P^a$ is also TP_2. Hence, we have $\pi P^a \geq_{mlr} \pi' P^a$. □

Lemma 5.4 If $\pi \geq_{mlr} \pi'$, then $\theta(\pi, a, k) \geq_{mlr} \theta(\pi', a, k)$.

Proof: From Lemma 5.3, we have $\pi P^a \geq_{mlr} \pi' P^a$, i.e.,

$$\begin{vmatrix} (\pi P^a)_i & (\pi' P^a)_i \\ (\pi P^a)_j & (\pi' P^a)_j \end{vmatrix} \geq 0 \text{ for all } i \leq j \text{ and } i, j \in S.$$

Hence,

$$\begin{vmatrix} \theta(i|\pi, a, k) & \theta(i|\pi', a, k) \\ \theta(j|\pi, a, k) & \theta(j|\pi', a, k) \end{vmatrix} = \begin{vmatrix} \dfrac{(\pi P^a)_i q_{ik}^a}{(\pi P^a Q^a)_k} & \dfrac{(\pi' P^a)_i q_{ik}^a}{(\pi' P^a Q^a)_k} \\ \dfrac{(\pi P^a)_j q_{jk}^a}{(\pi P^a Q^a)_k} & \dfrac{(\pi' P^a)_j q_{jk}^a}{(\pi' P^a Q^a)_k} \end{vmatrix}$$

$$= \frac{q_{ik}^a q_{jk}^a}{(\pi P^a Q^a)_k (\pi' P^a Q^a)_k} \begin{vmatrix} (\pi P^a)_i & (\pi' P^a)_i \\ (\pi P^a)_j & (\pi' P^a)_j \end{vmatrix} \geq 0$$

which completes the proof. □

Lemma 5.5 Under the assumption (2), we have $\theta(\pi, a, h) \geq_{mlr} \theta(\pi, a, k)$ for $h \leq k$ and $h, k \in O, a \in A - \{2, 3\}$.

Proof: With regard to the assumpption that Q^a is a TP_2 matrix for $i \le j$ and $h \le k$, it follows directly from definition (2.3),

$$\begin{vmatrix} \theta(i|\pi,a,h) & \theta(i|\pi,a,k) \\ \theta(j|\pi,a,h) & \theta(j|\pi,a,k) \end{vmatrix} = \begin{vmatrix} \frac{(\pi P^a)_i q^a_{ih}}{(\pi P^a Q^a)_h} & \frac{(\pi P^a)_i q^a_{ik}}{(\pi P^a Q^a)_k} \\ \frac{(\pi P^a)_j q^a_{jh}}{(\pi P^a Q^a)_h} & \frac{(\pi P^a)_j q^a_{jk}}{(\pi P^a Q^a)_k} \end{vmatrix}$$

$$= \frac{(\pi P^a)_i (\pi P^a)_j}{(\pi P^a Q^a)_h (\pi P^a Q^a)_k} \begin{vmatrix} q^a_{ih} & q^a_{ik} \\ q^a_{jh} & q^a_{jk} \end{vmatrix} \ge 0$$

This is equivalent to $\theta(\pi,a,h) \ge_{mlr} \theta(\pi,a,k)$. □

Lemma 5.6 Under assumptions (1)-(2), if $\pi \ge_{mlr} \pi'$, then $\sigma(\pi,a) \ge_{mlr} \sigma(\pi',a)$.

Proof: Using Lemma 5.2, we know that $\pi P^a Q^a \ge_{mlr} \pi' P^a Q^a$, i.e. $\sigma(\pi,a) \ge_{mlr} \sigma(\pi',a)$. □

In order to demonstrate the characteristics of the optimal value function $v(\pi)$, we let f be a real-valued function on R^n such that $f : S \to R^n \in F^n$.

Lemma 5.7 If $\pi \ge_{mlr} \pi'$, then $\pi f \ge \pi' f$.

Proof: Applying Lemma 5.1 and Derman's (1963) IFR property, we derive the result straightforwardly. □

According to assumptions (1)-(4), it is a fact that $v(\pi)$ is monotonically nondecreasing in π with respect to $\ge_{mlr}$ partial order.

Lemma 5.8 If $\pi \ge_{mlr} \pi'$, then $v(\pi) \ge v(\pi')$.

Proof: Since R, $-M^q$, and $-C \in F^n$, we can use Lemmas 5.4-5.7 and recursion (4.2) to conclude that

$$v(\pi) = \max_{a \in A}\{\Phi(\pi,a,v)\}$$

$$
\begin{aligned}
&= \max \begin{cases} \pi r + \beta \sum_{k=1}^{\omega} \sigma(k|\pi, 0) v\Big(\theta(\pi, 0, k)\Big) \\ \pi r - I + \beta \sum_{i=1}^{n} (\pi P^0)_i v(e^i) \\ \pi(-M^q) + r_q + \beta \sum_{k=1}^{\omega} \sigma(k|\pi, 2) v\Big(\theta(\pi, 2, k)\Big) \\ \pi(-C) + r_1 + \beta \sum_{k=1}^{\omega} \sigma(k|\pi, 3) v\Big(\theta(\pi, 3, k)\Big) \end{cases} \\
&\geq \max \begin{cases} \pi' r + \beta \sum_{k=1}^{\omega} \sigma(k|\pi', 0) v\Big(\theta(\pi', 0, k)\Big) \\ \pi' r - I + \beta \sum_{i=1}^{n} (\pi' P^0)_i v(e^i) \\ \pi'(-M^q) + r_q + \beta \sum_{k=1}^{\omega} \sigma(k|\pi', 2) v\Big(\theta(\pi', 2, k)\Big) \\ \pi'(-C) + r_1 + \beta \sum_{k=1}^{\omega} \sigma(k|\pi', 3) v\Big(\theta(\pi', 3, k)\Big) \end{cases} \\
&= v(\pi')
\end{aligned}
$$

for all $\pi, \pi' \in \Big(\prod(S), \geq_{mlr}\Big)$ □

The following Lemmas provide sufficient conditions for the main structural results.

Lemma 5.9 Under the assumptions (1)-(7), if $\pi \geq_{mlr} \pi'$, then

a) $\Phi(\pi, 0, v) - \Phi(\pi, 2, v) \geq \Phi(\pi', 0, v) - \Phi(\pi', 2, v)$;

b) $\Phi(\pi, 0, v) - \Phi(\pi, 3, v) \geq \Phi(\pi', 0, v) - \Phi(\pi', 3, v)$;

c) $\Phi(\pi, 1, v) - \Phi(\pi, 2, v) \geq \Phi(\pi', 1, v) - \Phi(\pi', 2, v)$;

d) $\Phi(\pi, 1, v) - \Phi(\pi, 3, v) \geq \Phi(\pi', 1, v) - \Phi(\pi', 3, v)$;

e) $\Phi(\pi, 2, v) - \Phi(\pi, 3, v) \geq \Phi(\pi', 2, v) - \Phi(\pi', 3, v)$.

Proof of (a): Based on assumptions (1)-(5), Lemmas 5.4 and 5.6–5.8, we know that

$$
\begin{aligned}
&\Phi(\pi, 0, v) - \Phi(\pi, 2, v) \\
&= \pi(r + M^q) + \beta \left[\sum_{k=1}^{\omega} \sigma(k|\pi, 0) v\Big(\theta(\pi, 0, k)\Big) - \sum_{k=1}^{\omega} \sigma(k|e^q, 0) v\Big(\theta(e^q, 0, k)\Big) \right]
\end{aligned}
$$

$$\geq \pi'(r + M^q) + \beta\left[\sum_{k=1}^{\omega}\sigma(k|\pi',0)v\Big(\theta(\pi',0,k)\Big) - \sum_{k=1}^{\omega}\sigma(k|e^q,0)v\Big(\theta(e^q,0,k)\Big)\right]$$
$$= \Phi(\pi',0,v) - \Phi(\pi',2,v)$$

Analogously, we can prove the remaining results. □

By using definition (2.4), Lemma 5.9 reveals that $\Phi(\pi, a, v)$ has antitone differences on

$$\Big(\prod(S), \geq_{mlr}\Big) \times A^0, A^0 \equiv \{0,2\}; \quad \Big(\prod(S), \geq_{mlr}\Big) \times A^1, A^1 \equiv \{0,3\};$$
$$\Big(\prod(S), \geq_{mlr}\Big) \times A^2, A^2 \equiv \{1,2\}; \quad \Big(\prod(S), \geq_{mlr}\Big) \times A^3, A^3 \equiv \{1,3\};$$
$$\Big(\prod(S), \geq_{mlr}\Big) \times A^4, A^4 \equiv \{2,3\} \text{ respectively.}$$

Hence, there exists an optimal policy which satisfies Lemma 5.9, monotonically divided into at most four regions, no action, inspection, repair, and replacement. It means that the more deteriorated the state of the system is , the higher the action is taken.

As in the previous lemma, by assumptions (1)-(6) and (8), the following lemma is yielded.

Lemma 5.10 If $\pi \geq_{mlr} \pi'$ then

a) $\Phi(\pi,0,v) - \Phi(\pi,2,v) \geq \Phi(\pi',0,v) - \Phi(\pi',2,v)$;

b) $\Phi(\pi,0,v) - \Phi(\pi,3,v) \geq \Phi(\pi',0,v) - \Phi(\pi',3,v)$;

c) $\Phi(\pi,1,v) - \Phi(\pi,2,v) \geq \Phi(\pi',1,v) - \Phi(\pi',2,v)$;

d) $\Phi(\pi,1,v) - \Phi(\pi,3,v) \geq \Phi(\pi',1,v) - \Phi(\pi',3,v)$;

e) $\Phi(\pi,2,v) - \Phi(\pi,3,v) \leq \Phi(\pi',2,v) - \Phi(\pi',3,v)$.

Proof: It is similar to Lemma 5.9 and thus is omitted. □

Lemma 5.10 states that $\Phi(\pi, a, v)$ has antitone differences on

$$\Big(\prod(S), \geq_{mlr}\Big) \times A^0, A^0 \equiv \{0,2\}; \quad \Big(\prod(S), \geq_{mlr}\Big) \times A^1, A^1 \equiv \{0,3\};$$
$$\Big(\prod(S), \geq_{mlr}\Big) \times A^2, A^2 \equiv \{1,2\}; \quad \Big(\prod(S), \geq_{mlr}\Big) \times A^3, A^3 \equiv \{1,3\} \text{ respectively.}$$

However, $\left(\prod(S), \geq_{mlr}\right)$ has isotone differences on $A^4 \equiv \{3, 2\}$.

Obviously, the optimal policy satisfying Lemma 5.10 is divided at most four regions (namely no action, inspection, replacement, and repair), but differs from the previous lemma in the replacement and repair order. We see that the more deteriorated the state of the system is, the higher the action is taken on A^0, A^1, A^2, and A^3. The more deteriorated the state of the system is, the lower the action is taken on A^4.

Lemma 5.11 $\Phi(\pi, a, v)$ ia an affine function for $a \in A - \{0\}$.
Proof: Let $\pi = (1-\lambda)\pi^1 + \lambda\pi^2$ for $\lambda \in$[0,1] and $\pi^1, \pi^2 \in \left(\prod(S), \geq_{mlr}\right)$.
From recursion (4.3), we obtain

$$\begin{aligned}\Phi(\pi, 1, v) &= \pi r - I + \beta\sum_{i=1}^{n}(\pi P^0)_i v(e^i)\\ &= \Big[(1-\lambda)\pi^1 + \lambda\pi^2\Big]r - \Big[(1-\lambda)I + \lambda I\Big]\\ &\quad +(1-\lambda)\beta\sum_{i=1}^{n}(\pi^1 P^0)_i v(e^i) + \lambda\beta\sum_{i=1}^{n}(\pi^2 P^0)_i v(e^i)\\ &= (1-\lambda)\Phi(\pi^1, 1, v) + \lambda\Phi(\pi^2, 1, v)\end{aligned}$$

By the same token, the following equations can be derived.

$$\begin{aligned}\Phi(\pi, 2, v) &= (1-\lambda)\Phi(\pi^1, 2, v) + \lambda\Phi(\pi^2, 2, v),\\ \Phi(\pi, 3, v) &= (1-\lambda)\Phi(\pi^1, 3, v) + \lambda\Phi(\pi^2, 3, v). \quad \square\end{aligned}$$

Lamma 5.12 $\Phi(\pi, 0, v)$ is a piecewise linear convex function.
Proof:

i) Let $\pi^1, \pi^2 \in \left(\prod(S), \geq_{mlr}\right)$ and $\pi = (1-\lambda)\pi^1 + \lambda\pi^2$ for $\lambda \in [0, 1]$. From recursion (4.1), consider the case under the finite horizon T, we have $\Phi(\pi, 0, v_T^T) = \pi r$. πr is a piecewise linear convex function.

ii) Suppose $\Phi(\pi, 0, v_{t+1}^T)$ is a convex function, then $v_t^T(\pi)$ is also convex.

iii) For $t < T-1$, and since $\dfrac{(1-\lambda)(\pi^1 P^0 Q^0)_k + \lambda(\pi^2 P^0 Q^0)_k}{(\pi P^0 Q^0)_k} = 1$, we get

$$
\begin{aligned}
\Phi(\pi.0, v_t^T) &= \pi r + \beta \sum_{k=1}^{\omega} \sigma(k|\pi, 0) v_t^T\Big(\theta(\pi, 0, k)\Big) \\
&= \pi r + \beta \sum_{k=1}^{\omega} (\pi P^0 Q^0)_k v_t^T \left(\frac{\Big((\pi P^0)_j q_{jk}^0\Big)_{j\in S}}{(\pi P^0 Q^0)_k} \right) \\
&= \pi r + \beta \sum_{k=1}^{\omega} (\pi P^0 Q^0)_k v_t^T \left(\frac{(1-\lambda)(\pi^1 P^0 Q^0)_k \Big((\pi^1 P^0)_j q_{jk}^0\Big)_{j\in S}}{(\pi P^0 Q^0)_k (\pi^1 P^0 Q^0)_k} \right. \\
&\quad \left. + \frac{\lambda(\pi^2 P^0 Q^0)_k \Big((\pi^2 P^0)_j q_{jk}^0\Big)_{j\in S}}{(\pi P^0 Q^0)_k (\pi^2 P^0 Q^0)_k} \right) \\
&\le \pi r + (1-\lambda)\beta \sum_{k=1}^{\omega} (\pi^1 P^0 Q^0)_k v_t^T \left(\Big(\frac{(\pi^1 P^0)_j q_{jk}^0}{\pi^1 P^0 Q^0)_k} \Big)_{j\in S} \right) \\
&\quad + \lambda\beta \sum_{k=1}^{\omega} (\pi^2 P^0 Q^0)_k v_t^T \left(\Big(\frac{(\pi^2 P^0)_j q_{jk}^0}{\pi^2 P^0 Q^0)_k} \Big)_{j\in S} \right) \\
&= (1-\lambda)\Phi_{t-1}(\pi, 0, v_t^T) + \lambda \Phi_{t-1}(\pi, 0, v_t^T)
\end{aligned}
$$

where

$$
\Big((\pi P^0)_j q_{jk}^0\Big)_{j\in S} = \Big((\pi P^0)_1 q_{1k}^0, (\pi P^0)_2 q_{2k}^0, \ldots, (\pi P^0)_n q_{nk}^0\Big).
$$

The result then follows by induction. □

Above Lemmas 5.11 and 5.12 initiate the following:

Lemma 5.13

a) $\Phi(\pi, a, v) - \Phi(\pi, 1, v)$ are affine functions for $a \in A - \{1\}$.

b) $\Phi(\pi, 0, v) - \Phi(\pi, 1, v)$ is a convex function.

In addition to characterize the structural results of optimal inspection-repair-replacement policy, we define the subsets of $\Big(\prod(S), \ge_{mlr}\Big)$ for each action:

$$
D_a = \{\pi \in \Big(\prod(S), \ge_{mlr}\Big) : v(\pi) = \Phi(\pi, a, v)\} \text{ for } a \in A.
$$

Lemma 5.14 D_1 is a convex subset of $\left(\prod(S), \geq_{mlr}\right)$.

Proof: According to the definition of D_a, we know that if $\pi \in D_1$ and $a \in A - \{1\}$, then $\Phi(\pi, a, v) - \Phi(\pi, 1, v) \leq 0$ for all $a \in A - \{1\}$. Similarly, for $\pi' \in D_1, \Phi(\pi', a, v) - \Phi(\pi', a, v) \leq 0$. Provided the above and Lemma 5.13, these inequalities hold:

$$\Phi\Big((1-\lambda)\pi + \lambda\pi', a, v\Big) - \Phi\Big((1-\lambda)\pi + \lambda\pi', 1, v\Big) \leq 0 \text{ for all } a \in A - \{1\}.$$

The proof is therefore done. □

From Lemmas 5.9, 5.10 and 5.14, we arrive at the main structural results:

Theorem 5.1 (At most five-region policy I):

Under assumptions (1)-(7), if $\pi^1 \geq_{mlr} \pi^2$, then there at most exist $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4, 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq 1$ such that

$$\begin{aligned}
D_0 &\supset \left[\pi^1, (1-\lambda_1)\pi^1 + \lambda_1\pi^2\right] \\
D_1 &\supset \left[(1-\lambda_1)\pi^1 + \lambda_1\pi^2, (1-\lambda_2)\pi^1 + \lambda_2\pi^2\right] \\
D_0 &\supset \left[(1-\lambda_2)\pi^1 + \lambda_2\pi^2, (1-\lambda_3)\pi^1 + \lambda_3\pi^2\right] \\
D_3 &\supset \left[(1-\lambda_3)\pi^1 + \lambda_3\pi^2, (1-\lambda_4)\pi^1 + \lambda_4\pi^2\right] \\
D_2 &\supset \left[(1-\lambda_4)\pi^1 + \lambda_4\pi^2, \pi^2\right]
\end{aligned}$$

where $[\pi^1, \pi^2] \equiv \{\pi \in R^n : \pi = (1-\lambda)\pi^1 + \lambda\pi^2, \lambda \in [0,1]\}$.

Above theorem verifies that the optimal policy has at most five control regions which are generated by $\lambda_1, \lambda_2, \lambda_3$, and λ_4. It is clear that Lemma 5.14 guarantees an inspection region in between two no action regions.

Theorem 5.2 (At most five-region policy II):

Under assumptions (1)-(6) and (8), if $\pi^1 \geq_{mlr} \pi^2$, then there at most exist $\lambda_1, \lambda_2, \lambda_3$ and λ_4, $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq 1$ such that

$$D_0 \supset \left[\pi^1, (1-\lambda_1)\pi^1 + \lambda_1\pi^2\right]$$

$$D_1 \supset \left[(1-\lambda_1)\pi^1 + \lambda_1\pi^2, (1-\lambda_2)\pi^1 + \lambda_2\pi^2\right]$$
$$D_0 \supset \left[(1-\lambda_2)\pi^1 + \lambda_2\pi^2, (1-\lambda_3)\pi^1 + \lambda_3\pi^2\right]$$
$$D_3 \supset \left[(1-\lambda_3)\pi^1 + \lambda_3\pi^2, (1-\lambda_4)\pi^1 + \lambda_4\pi^2\right]$$
$$D_2 \supset \left[(1-\lambda_4)\pi^1 + \lambda_4\pi^2, \pi^2\right]$$

where $[\pi^1, \pi^2] \equiv \{\pi \in R^n : \pi = (1-\lambda)\pi^1 + \lambda\pi^2, \lambda \in [0,1]\}$.

Proof: By Lemmas 5.10 and 5.14, we obtain the results straightforwardly. □

Comparing Theorem 5.1 with the above theorem, the optimal policy structures indicate that replacement and repair regions are interchangeable. In view of management applications, Theorem 5.1 is applicable to general systems where the replacement cost is cheaper than repair cost during the worse state; however, under assumption(8), Theorem 5.2 is applicable to costly systems, e.g., military defense systems. Such systems usually are difficult to obtain and its replacement costs are to high, especially from a certain worse state on, the gaps between repair and replacement costs tend to narrow. Thus the repair action is always a preferred alternative.

VI. ALTERNATE MODELING RESULTS

The above results focus on an inspection-repair-replacement model in which all actions are taken at beginning, while the repair and replacement costs depend on the state probability vector and reward earned relevant to the improved state. In order to get more understanding of the porblem, we probe into the below two different models.

First, we consider the case where the actions compose of terms for "no action, instantaneous inspection and repair at end, instantaneous replacement at beginning," Obviously, the inspection cost I must be discounted, and the expected repair cost differs from that of recursion (4.3). Hence, the recursion formula can be expressed as:

$$
\begin{aligned}
v(\pi) &= \max_{a\in A}\{\Phi(\pi,a,v)\} \\
&= \max\begin{cases}
\pi r+\beta\sum_{k=1}^{\omega}\sigma(k|\pi,0)v\Big(\theta(\pi,0,k)\Big) & ,\text{a=0}\\
\pi r+\beta\Big[-I+\sum_{i=1}^{n}(\pi P^0)_i v(e^i)\Big] & ,\text{a=1}\\
\pi r+\beta\Big\{\Big[\sum_{k=1}^{\omega}\sigma(k|\pi,0)\Big(\theta(\pi,0,k)\Big)\Big](-M^q)+v(e^q)\Big\} & ,\text{a=2}\\
\pi(-C)+r_1+\beta\sum_{k=1}^{\omega}\sigma(k|e^1,0)v\Big(\theta(e^1,0,k)\Big) & ,\text{a=3}
\end{cases}
\end{aligned} \tag{6.1}
$$

To characterize the structure of the optimal policy, the recursion (6.1) can be rewritten as follows:

$$
v(\pi)=\max\begin{cases}
\pi r+\beta\sum_{k=1}^{\omega}\sigma(k|\pi,0)v\Big(\theta(\pi,0,k)\Big) & ,\text{a=0}\\
\pi r+\beta\Big[-I+\sum_{i=1}^{n}(\pi P^0)_i v(e^i)\Big] & ,\text{a=1}\\
\pi r+\beta\Big\{\Big[\sum_{k=1}^{\omega}\Big((\pi P^0)_j q_{ik}^0\Big)_{j\in S}\Big](-M^q)+v(e^q)\Big\} & ,\text{a=2}\\
\pi(-C)+r_1+\beta\sum_{k=1}^{\omega}\sigma(k|e^1,0)v\Big(\theta(e^1,0,k)\Big) & ,\text{a=3}
\end{cases} \tag{6.2}
$$

It is straightforward to show the proofs of Lemmas 5.9 through 5.14 and the main Theorems 5.1-5.2 go through directly with recursion (6.2). Hence, structural results remain the same.

Second, we treat the case where "no action, inspection at beginning, instantaneous repair and replacement at end." Under these conditions, the optimal policy can be constructed from the following recursion.

$$
v(\pi) = \max_{a\in A}\{\Phi(\pi,a,v)\}
$$

$$
= \max \begin{cases} \pi r + \beta \sum_{k=1}^{\omega} \sigma(k|\pi,0) v\Big(\theta(\pi,0,k)\Big) & ,a=0 \\ \pi r - I + \beta \sum_{i=1}^{n} (\pi P^0)_i v(e^i) & ,a=1 \\ \pi r + \beta \Big\{ \Big[\sum_{k=1}^{\omega} \sigma(k|\pi,0)\Big(\theta(\pi,0,k)\Big)\Big](-M^q) + v(e^q) \Big\} & ,a=2 \\ \pi r + \beta \Big\{ \Big[\sum_{k=1}^{\omega} \sigma(k|\pi,0)\Big(\theta(\pi,0,k)\Big)\Big](-C) + v(e^1) \Big\} & ,a=3 \end{cases} \tag{6.3}
$$

Similarly, it can be written as follows:

$$
v(\pi) = \max \begin{cases} \pi r + \beta \sum_{k=1}^{\omega} \sigma(k|\pi,0) v\Big(\theta(\pi,0,k)\Big) & ,a=0 \\ \pi r - I + \beta \sum_{i=1}^{n} (\pi P^0)_i v(e^i) & ,a=1 \\ \pi r + \beta \Big\{ \Big[\sum_{k=1}^{\omega} \Big((\pi P^0)_j q^0_{ik}\Big)_{j \in S} \Big](-M^q) + v(e^q) \Big\} & ,a=2 \\ \pi r + \beta \Big\{ \Big[\sum_{k=1}^{\omega} \Big((\pi P^0)_j q^0_{jk}\Big)_{j \in S} \Big](-C) + v(e^1) \Big\} & ,a=3 \end{cases} \tag{6.4}
$$

Comparing recursions (6.4) and (6.2), we see that their difference occurs at inspection and replacement, however, recursions (6.4) and (4.3) hold the difference at repair and replacement. From here, we note that the common difference between (6.4) and (6.2) on the one hand and (6.4) and (4.3) on the other hand is the effect of replacement action, though this common difference does not affect the proofs of the main structural results in this paper.

VII. CONCLUSION AND FURTHER STUDIES

In this paper, we have modelled the discrete-time, infinite horizon for Markov inspection-repair-replacement problem which is partially observable. Using partial orders of $\geq_{st}$ and $\geq_{mlr}$, and TP_2 to identify the preferred ranks for the state probability vectors of the system deterioration. Moreover, under several reasonable conditions and relevant properties (such as isotone and antitone differences), in which the structural results are developed and summarized as follows.

1. Based on assumptions (1)-(7), for "no action, inspection at beginning, instantaneous repair and replacement at beginning," we have shown that the optimal policy has at most five-region structure. These five regions are "no action, inspection, no action, repair, and replacement."
2. Under assumptions (1)-(6) and (8), we get the different results, in which the regions of repair and replacement interchange.
3. Alternative models of "no action, instantaneous inspection and repair at end, instantaneous replacement at beginning," and "no action, inspection at beginning, instantaneous repair and replacement at end," are presented, and the main results obtained in 1 and 2 remain the same.

The further research includes:

- Developing the numerical techniques for solving POMDP'S.
- Studying on the semi-Markov decision processes (SMDP'S) with partial observation under the above models.
- Considering continuous time for POMDP'S.
- Treating the inspection-repair-replacement problem as the state-independent of inspection cost and taking the probability of success in a precise state into consideration.
- Searching for the alternative of optimal policy structure if there exists a statistical type I or type II errors for the imperfect inspection.

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